§ 5.4* Sequential Decoding

- Introduction
  - Sequential decoding [1] is a suboptimal decoding approach elaborated over the binary tree, circumventing the high complexity in Viterbi decoding of conv. codes with a large memory *m*.
  - By setting a dynamic threshold adjusted during the decoding, it compares different paths and selects the most likely one. Meanwhile, it minimizes the unnecessary node search.
  - For practical use, code rate must be strictly below channel capacity.
  - Fano decoding [2] is a depth-first-search (DFS) algorithm. By moving forward or backward over the tree, it aims to find the most likely path. In order to rationalize the decoding effort, a complexity budget can be imposed. That says the decoding will be terminated once the budget is reached.

[1] J. M. Wozencraft, "Sequential decoding for reliable communication," *Res. Lab. Elect.*, MIT, Cambridge, MA, USA, Tech. Rep. 325, Aug. 1957.

[2] R. M. Fano, "A heuristic discussion of probabilistic decoding," *IEEE Trans. Inf. Theory*, vol. IT-9, no. 2, pp. 64–74, Apr. 1963.

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- Let us denote codeword

$$\bar{c} = (c_1, c_2, \cdots, c_N) = ([\bar{c}]_t), \text{ where } t = 1,2,\cdots N,$$

and a received symbol vector

$$\bar{y} = (y_1, y_2, \cdots, y_N) = ([\bar{y}]_t), \text{ where } t = 1,2,\cdots N.$$

- MAP decoding yield codeword $\bar{c}_{i'}$, s.t.

$$P(\bar{c}_{i'}|\bar{y}) \geq P(\bar{c}_i|\bar{y}), \forall\, i \neq i'$$

- Sequential decoding aims to find a likely codeword $\bar{c}_{i'}$, s.t. its *a posteriori* probability $P(\bar{c}_{i'}|\bar{y})$ is greater than or equal to a threshold. Relaxing the MAP criterion, let $\sum_{\substack{i=1 \\ i\neq i'}}^{2^K} P(\bar{c}_i|\bar{y})$ be the threshold. The decoding finds $\bar{c}_{i'}$, s.t.

$$P(\bar{c}_{i'}|\bar{y}) \geq \sum_{i=1,\, i\neq i'}^{2^K} P(\bar{c}_i|\bar{y}) \tag{1}$$

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- Assumptions:
  Random coding,

$$P(\bar{c}_i) = \prod_{t=1}^{N} P([\bar{c}_i]_t) = 2^{-K}$$

  Discrete memoryless channel,

$$P(\bar{y}|\bar{c}_i) = \prod_{t=1}^{N} P([\bar{y}]_t | [\bar{c}_i]_t)$$

- By Bayes' rule, we have

$$P(\bar{y}|\bar{c}_i) = \frac{P(\bar{c}_i|\bar{y}) \cdot P(\bar{y})}{P(\bar{c}_i)}, \forall\ i$$

- The decoding yields a codeword $\bar{c}_{i'}$ that satisfies

$$P(\bar{y}|\bar{c}_{i'}) \geq \sum_{i=1,\ i\neq i'}^{2^K} P(\bar{y}|\bar{c}_i) \tag{2}$$

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- Let $\tilde{P}(\bar{y}|\bar{c}_i)$ denote the average of all transition probabilities $P(\bar{y}|\bar{c}_i)$ in the right hand side of (2).
- Hence

$$P(\bar{y}|\bar{c}_{i'}) \geq (2^K - 1) \cdot \tilde{P}(\bar{y}|\bar{c}_i), \tag{3}$$

where

$$\begin{aligned}\tilde{P}(\bar{y}|\bar{c}_i) &= \frac{\sum_{i=1}^{2^K-1} P(\bar{y}|\bar{c}_i)}{2^K - 1} \\ &\cong \sum_{i=1}^{2^K-1} P(\bar{y}|\bar{c}_i) \cdot P(\bar{c}_i) \\ &= \sum_{i=1}^{2^K-1} P(\bar{c}_i, \bar{y}) \\ &\cong P(\bar{y})\end{aligned}$$

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- Therefore, ineq. (3) becomes

$$P(\bar{y}|\bar{c}_{i'}) \geq (2^K - 1) \cdot P(\bar{y})$$

$$\Downarrow$$

$$\frac{P(\bar{y}|\bar{c}_{i'})}{P(\bar{y})} \geq 2^K - 1$$

$$\Downarrow$$

$$\prod_{t=1}^{N} \frac{P([\bar{y}]_t|[\bar{c}_{i'}]_t)}{P([\bar{y}]_t)} \geq 2^K - 1$$

$$\Downarrow \qquad \text{Approximate } 2^K - 1 \text{ to } 2^K$$

$$\log_2 \prod_{t=1}^{N} \frac{P([\bar{y}]_t|[\bar{c}_{i'}]_t)}{P([\bar{y}]_t)} \geq K$$

$$\Downarrow$$

$$\sum_{t=1}^{N} \left(\log_2 \frac{P([\bar{y}]_t|[\bar{c}_{i'}]_t)}{P([\bar{y}]_t)} - R \right) \geq 0$$

Mutual information $I([\bar{y}]_t; [\bar{c}_{i'}]_t)$

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- Denote $[\bar{c}]_t = \bar{c}_t$, $[\bar{y}]_t = \bar{y}_t$, Fano branch metric is

$$\lambda_t = \log_2 \frac{P(\bar{y}_t|\bar{c}_t)}{P(\bar{y}_t)} - n_0 \cdot R$$

where at time instant t, the input information is k_0 bits and the output codeword is n_0 bits. Code rate $R = \frac{k_0}{n_0}$.

- Metric of a path with length l is

$$\Lambda = \sum_{t=1}^{l} \lambda_t = \sum_{t=1}^{l} \log_2 \frac{P(\bar{y}_t|\bar{c}_t)}{P(\bar{y}_t)} - n_0 l R$$

- The path metric Λ increases if $\bar{y}_t$ can yield a right decision, and decreases vice versa.

- When comparing paths of different lengths, the path with the maximum path metric is considered the most likely one.

§ 5.4* Sequential Decoding

- **Fano decoding rules:**
  - **Initialization** (start from the root)**:** Let threshold $T = 0$, step size Δ (for adjusting T). E.g., let $\Delta = 4$.
  - **Computations** (over the code's binary tree): Calculate two forward nodes' path metrics, and denote them as $\Lambda_{\max}$ and $\Lambda_{\min}$, where $\Lambda_{\max} \geq \Lambda_{\min}$. Choose one of the below operations:
    - If $\Lambda_{\max} \geq T$, move forward to its child node and update path metric as $\Lambda_{\max}$;
    - If $\Lambda_{\max} < T$, move backward to the parent node. This may happen a few times until reaching an unexpended ancestral node with $\Lambda_{\min} > T$. The decoder then starts to computer from this node with a path metric of $\Lambda_{\min}$.
  - **Termination** (when reaches a leaf)**:** When decoder reach the leaf node, the decoding is completed and the path outputs yield the candidate codeword.
  - When reaching a node with path metric Λ, threshold T is **adjusted** as below:
    - If current node is reached from its parent node for the first time and $\Lambda > T + \Delta$, increase T by $T + \Delta \to T$, s.t. $\Lambda - \Delta < T \leq \Lambda$.
    - If current node is reached from its child node and $\Lambda < T$, decrease T by $T - \Delta \to T$, s.t. $\Lambda - \Delta < T \leq \Lambda$.
    - If root node is reached from its $\Lambda_{\min}$ child node, decrease T by $T - \Delta \to T$.

§ 5.4* Sequential Decoding

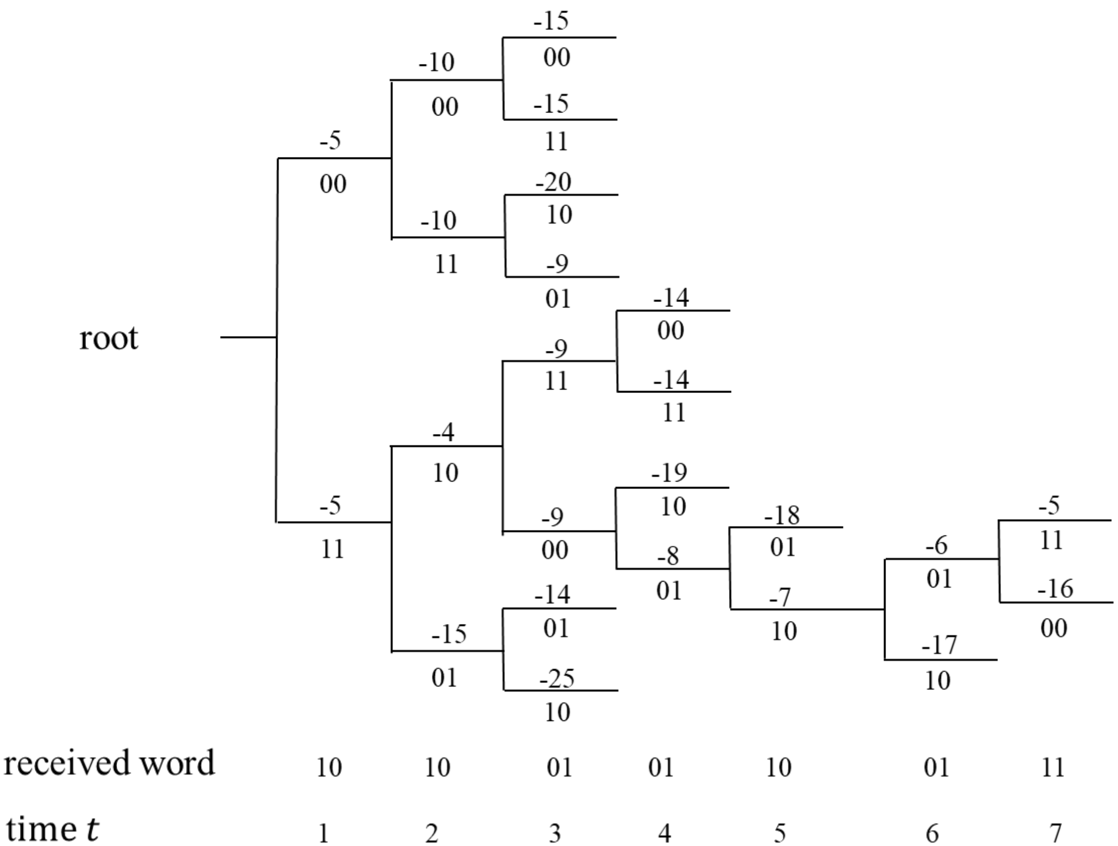
Example 5.7 Given the $(7, 5)_8$ conv. code, the message is $[m_1\, m_2\, m_3\, m_4\, m_5\, m_6\, m_7] =$ $[1\ 0\ 1\ 1\ 1\ 0\ 0]$, the codeword is $\left[c_1^1 c_1^2\ c_2^1 c_2^2\ c_3^1 c_3^2\ c_4^1 c_4^2\ c_5^1 c_5^2\ c_6^1 c_6^2\ c_7^1 c_7^2\right] =$ $[11\ 10\ 00\ 01\ 10\ 01\ 11]$, the received word is $\left[y_1^1 y_1^2\ y_2^1 y_2^2\ y_3^1 y_3^2\ y_4^1 y_4^2\ y_5^1 y_5^2\ y_6^1 y_6^2\ y_7^1 y_7^2\right] =$ $[10\ 10\ 01\ 01\ 10\ 01\ 11]$. Thus, $n_0 = 2$, $k_0 = 1$, $R = k_0/n_0 = 0.5$. Under BSC channel, $p = 0.02$, and with uniform input such that $P(y = 0) = P(y = 1) = 0.5$.

At time t, the branch metric λ_t between estimated message bit $\hat{m}_t$ or equivalently $\hat{c}_t^1 \hat{c}_t^2$ and received symbols $y_t^1 y_t^2$ only occurs three situations as

$$\lambda_t = \log_2 \frac{P(\bar{y}_t|\bar{c}_t)}{P(\bar{y}_t)} - n_0 \cdot R = \begin{cases} 2 \cdot \log_2 \dfrac{1-p}{0.5} - 2 \cdot R \approx 0.94 \approx 1 & \text{, no error} \\ \log_2 \dfrac{1-p}{0.5} + \log_2 \dfrac{p}{0.5} - 2 \cdot R \approx -4.67 \approx -5 & \text{, 1 error} \\ 2 \cdot \log_2 \dfrac{p}{0.5} - 2 \cdot R \approx -10.29 \approx -10 & \text{, 2 errors} \end{cases}$$

§ 5.4* Sequential Decoding

The following graph shows the outcome if all paths have been explored



Initialization

$\underline{T = 0}$, $\Delta = 4$

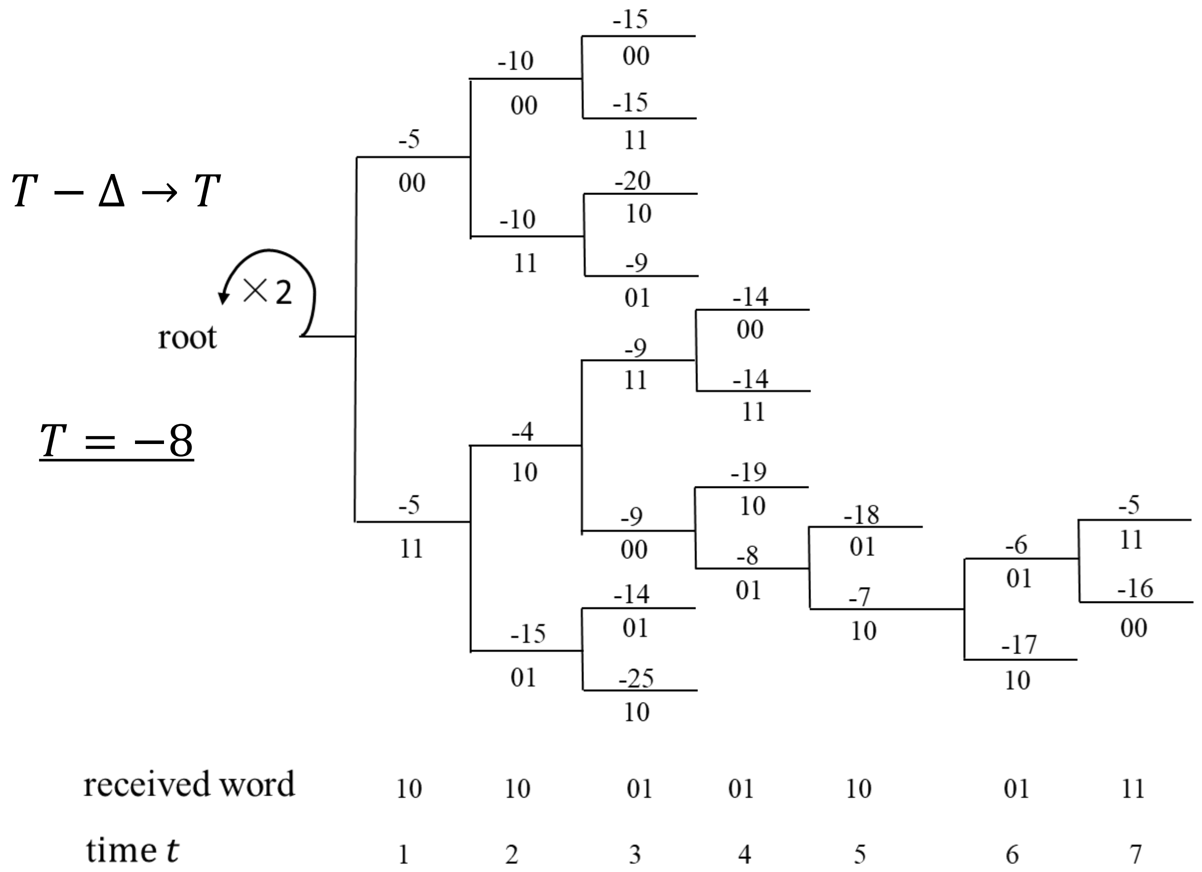
↑ Input bit as 0

↓ Input bit as 1

$$\frac{\text{Path metric}}{\hat{c}_t^1 \hat{c}_t^2}$$

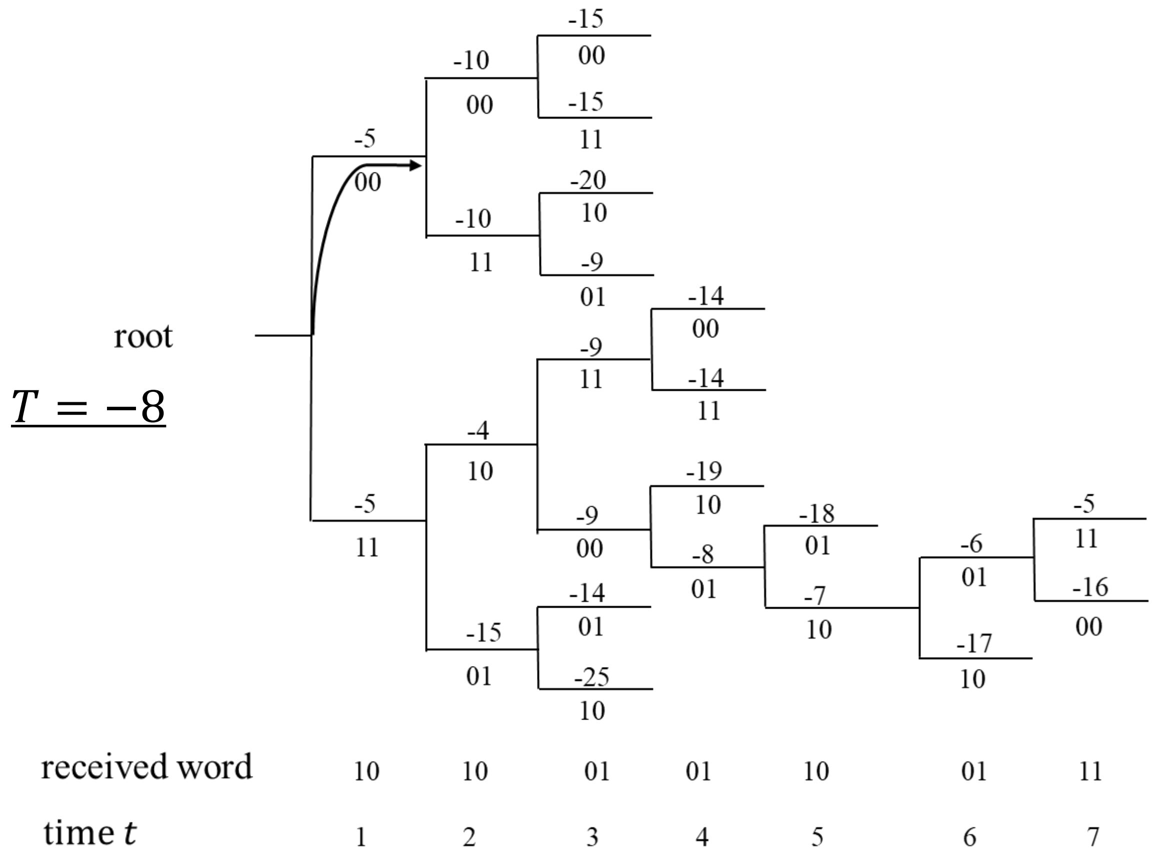
§ 5.4* Sequential Decoding

Calculate the path metric from the root node to the children nodes, we have $\Lambda_{\max} = -5$ and $\Lambda_{\max} < T$. Reduce threshold twice until decoder could depart from the root.



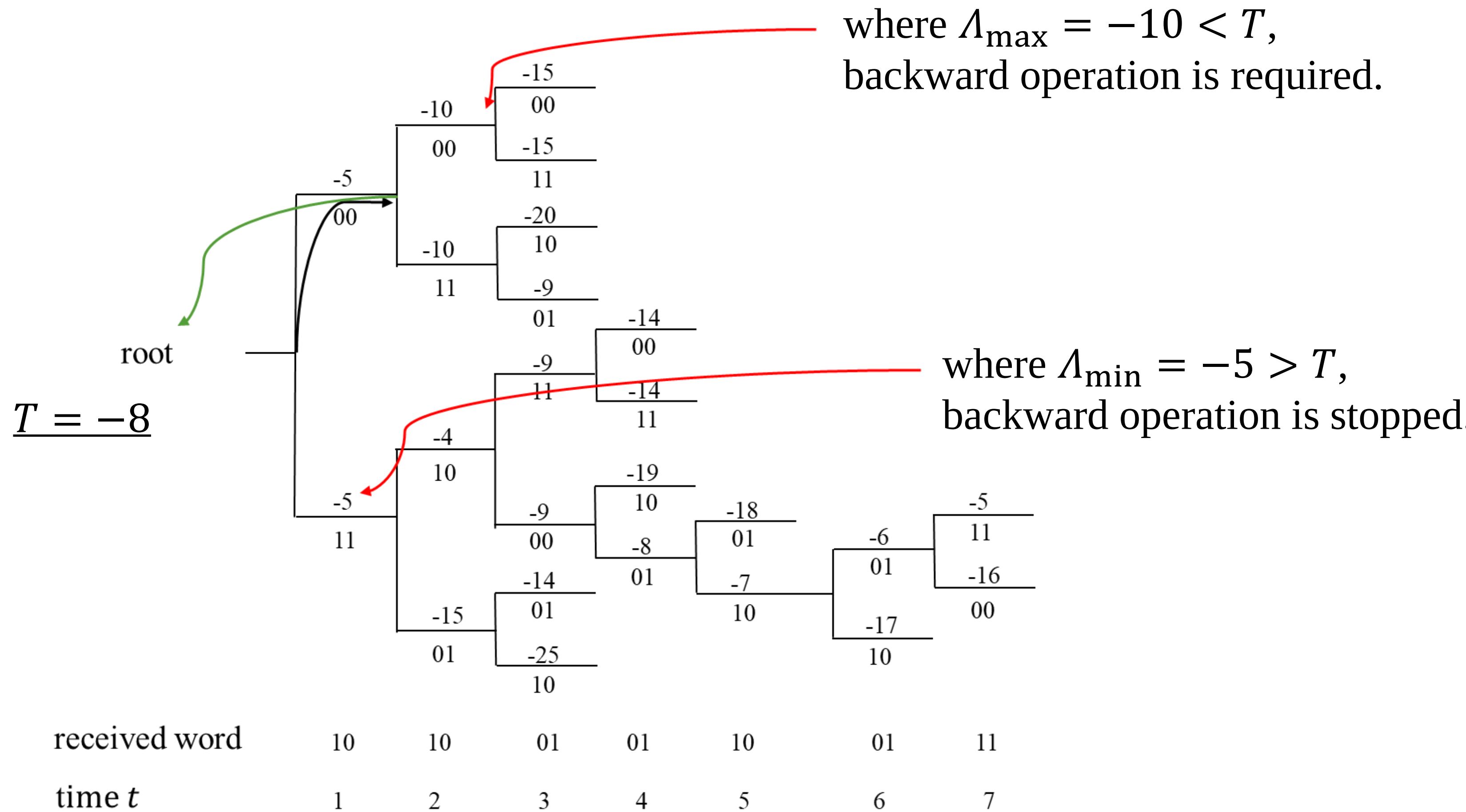
§ 5.4* Sequential Decoding

Move forward until $\Lambda_{\max} < T$.



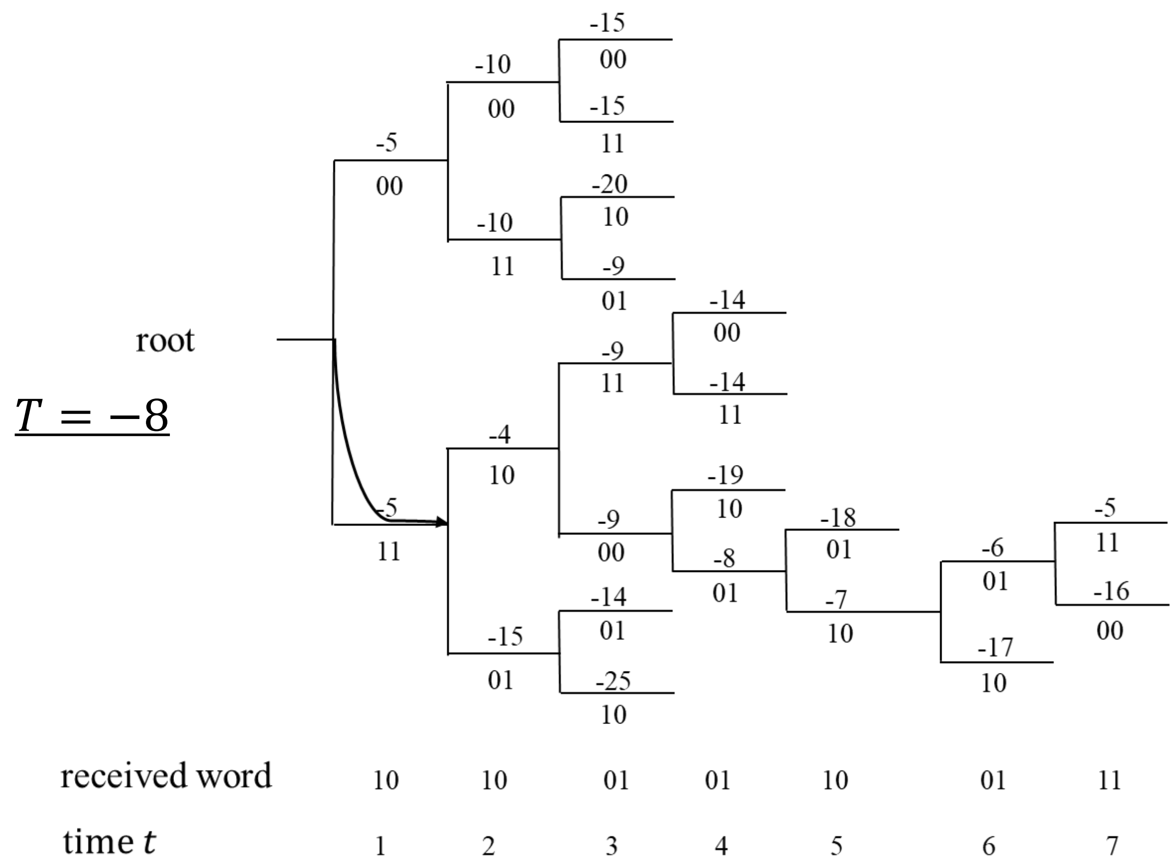
§ 5.4* Sequential Decoding

Move backward until an unexpended node with path metric $\Lambda_{\min} > T$.



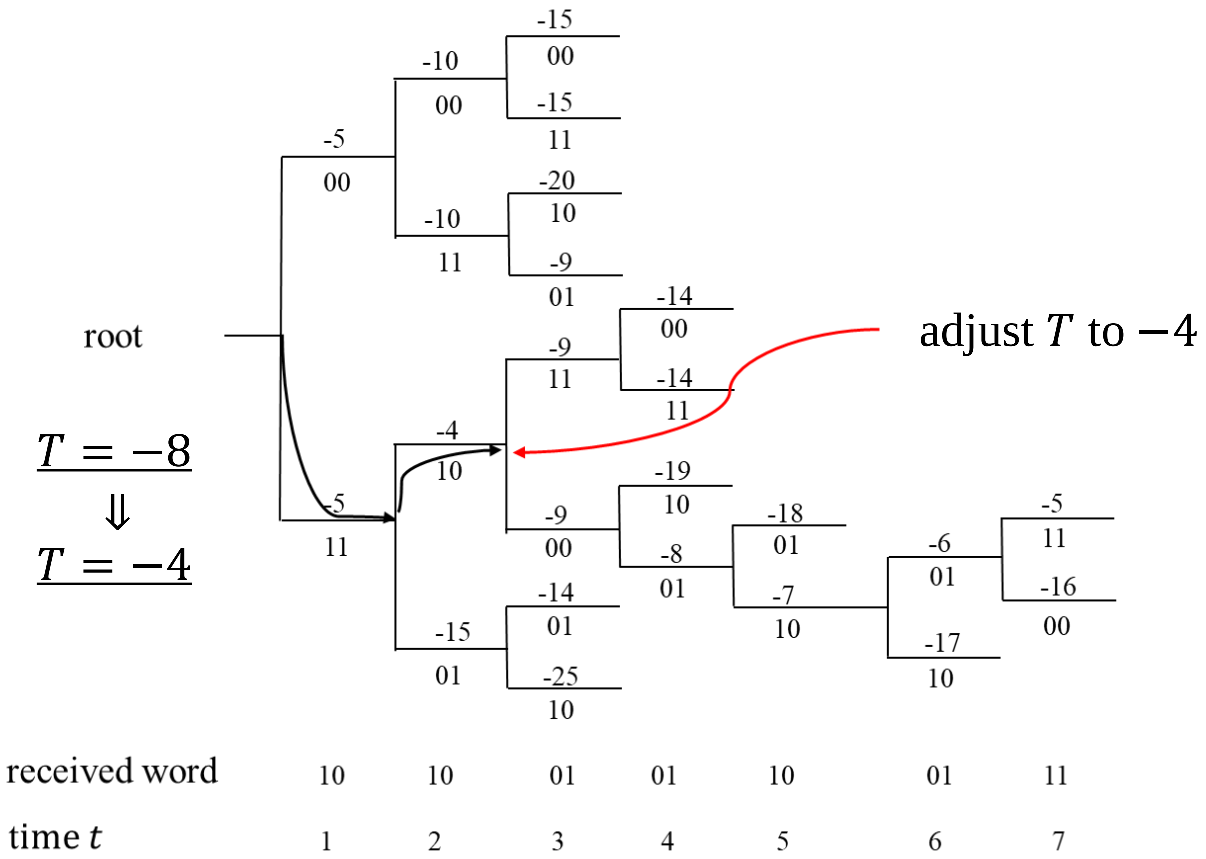
§ 5.4* Sequential Decoding

Move forward from node with path metric $\Lambda_{\min} = -5 > T$.



§ 5.4* Sequential Decoding

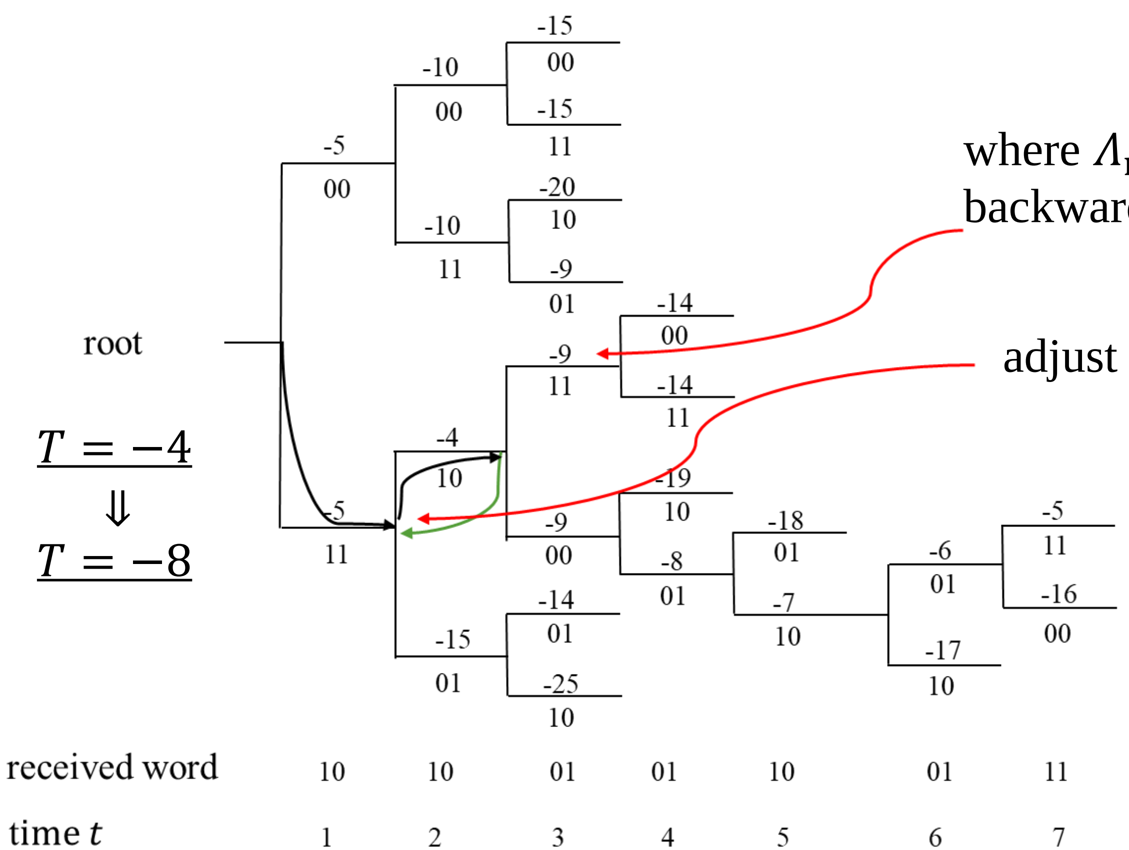
Move forward. If reach a node for the first time and its path metric $\Lambda = -4 \geq T + \Delta = -4$, let $T + \Delta \rightarrow T$, s.t. $\Lambda - \Delta < T \leq \Lambda$.



§ 5.4* Sequential Decoding

Move backward because $\Lambda_{\max} < T$.

Then, it is found that $\Lambda = -5 < T = -4$. Thus, Let $T - \Delta \to T$, s.t. $\Lambda - \Delta < T \leq \Lambda$.

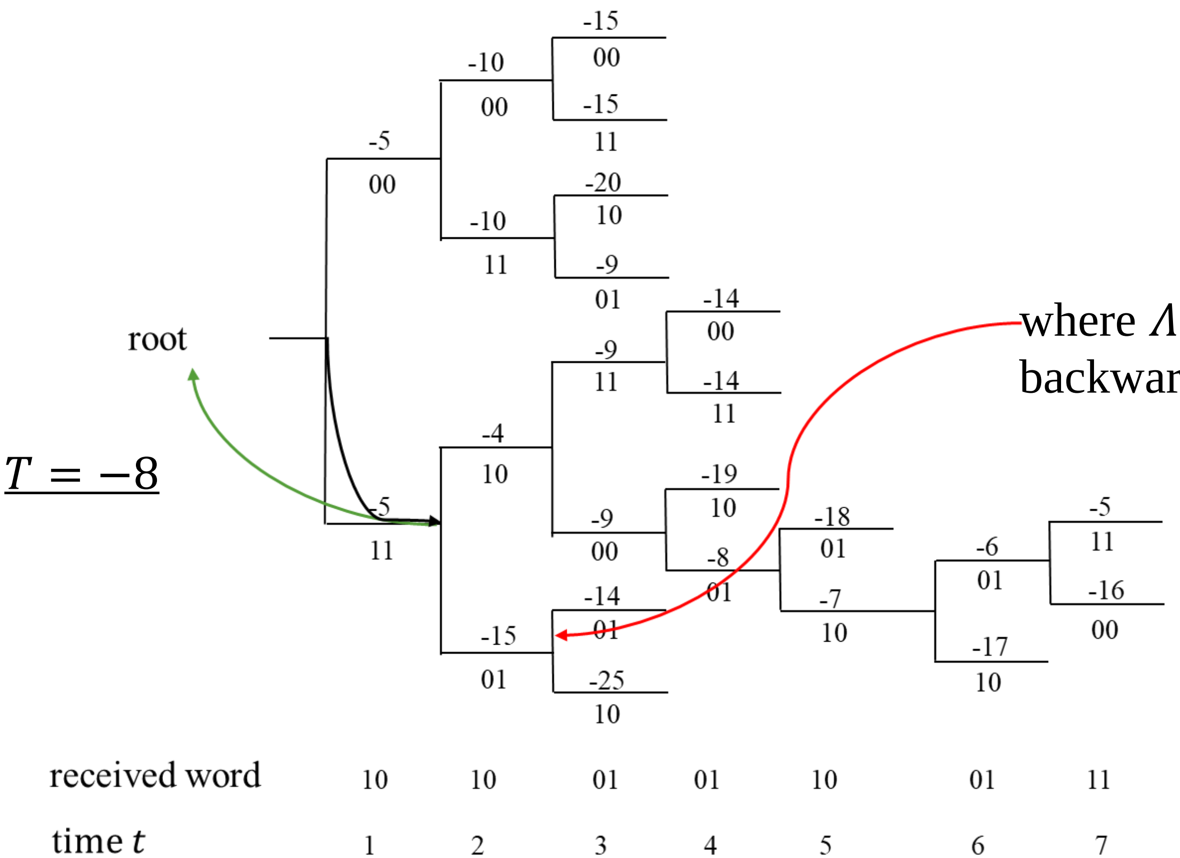


where $\Lambda_{\max} = -9 < T = -4$, backward operation is required.

adjust T to -8

§ 5.4* Sequential Decoding

Keep moving backward until an unexpended node with $\Lambda_{\min} > T$.



where $\Lambda_{\min} = -15 < T$,
backward operation is required.

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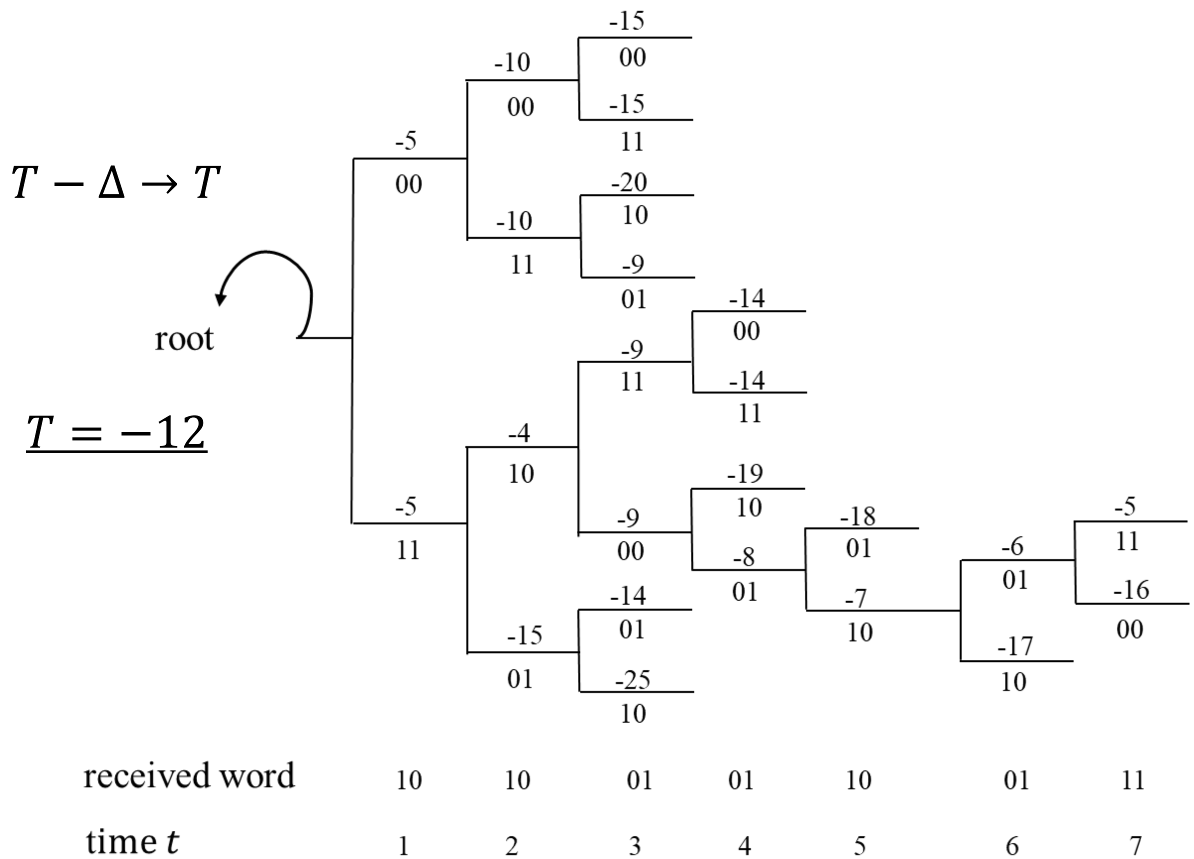
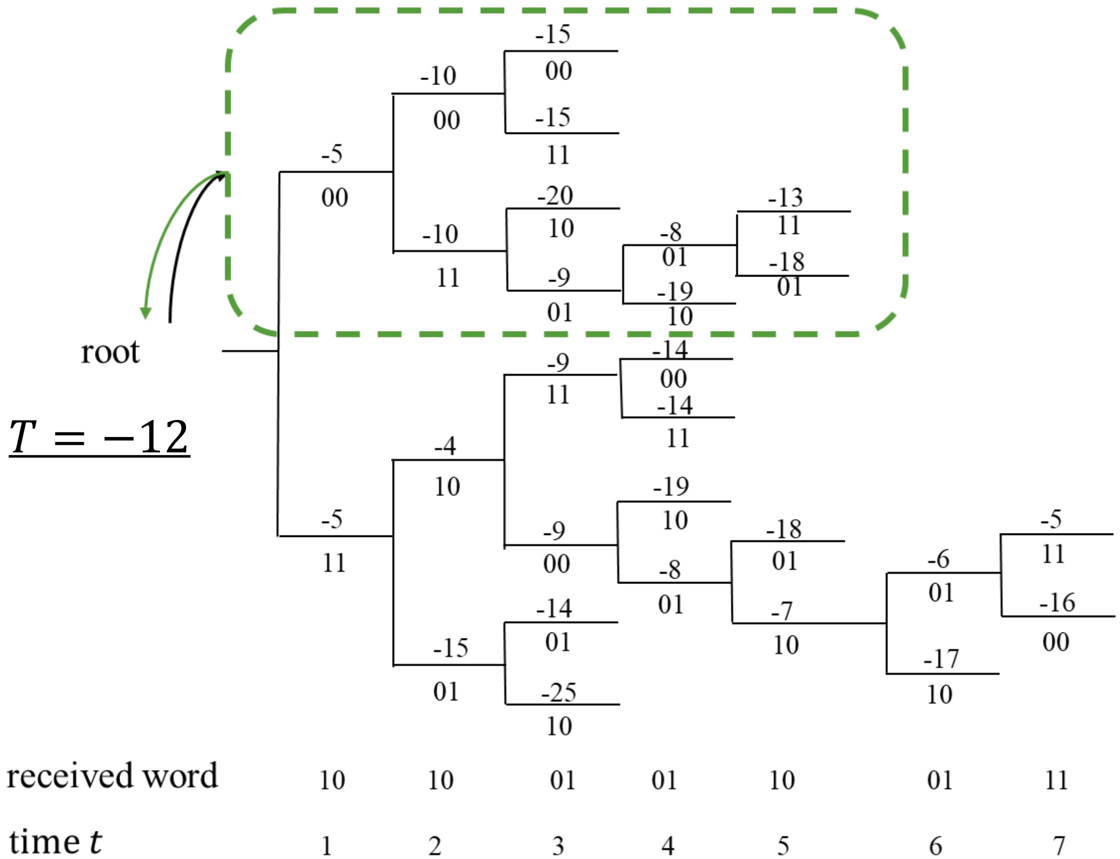
Reduce T by $T - \Delta \rightarrow T$ because root node is reached from its $\Lambda_{\min}$ child node.



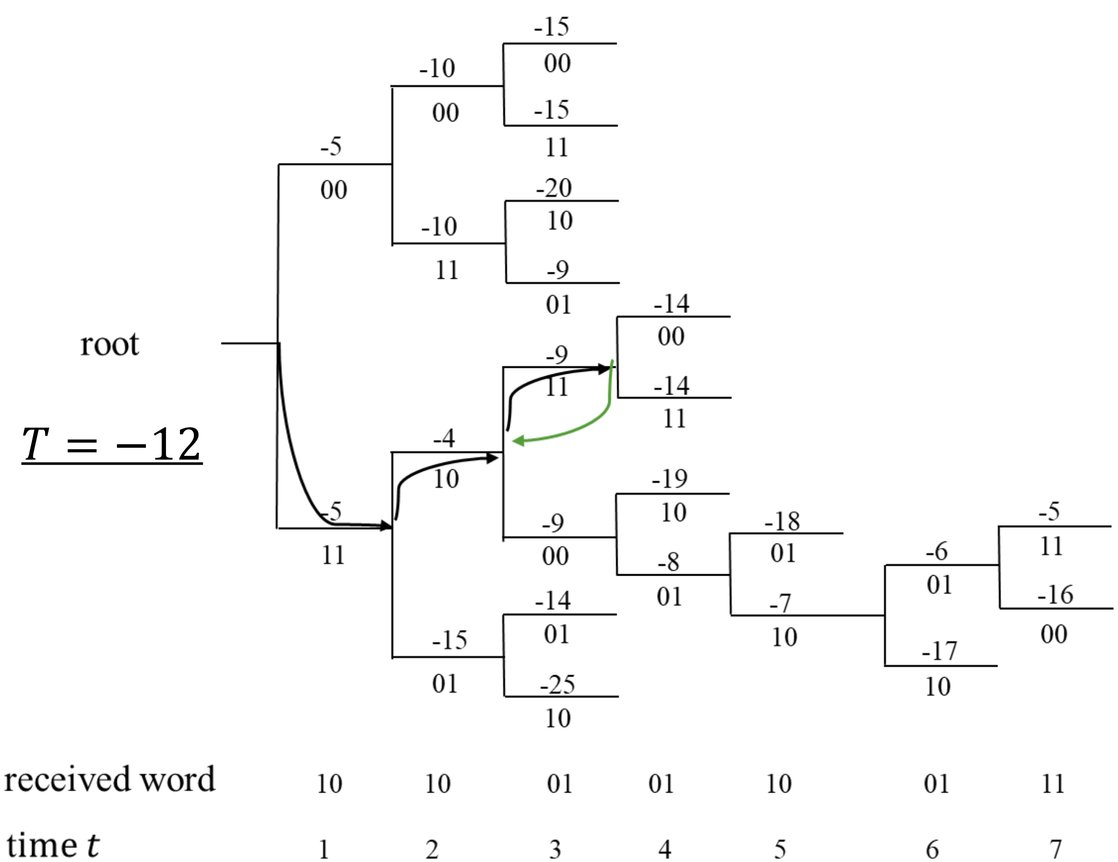
§ 5.4* Sequential Decoding

Exploration of the framed (by green dash) paths cannot reach the leaf.



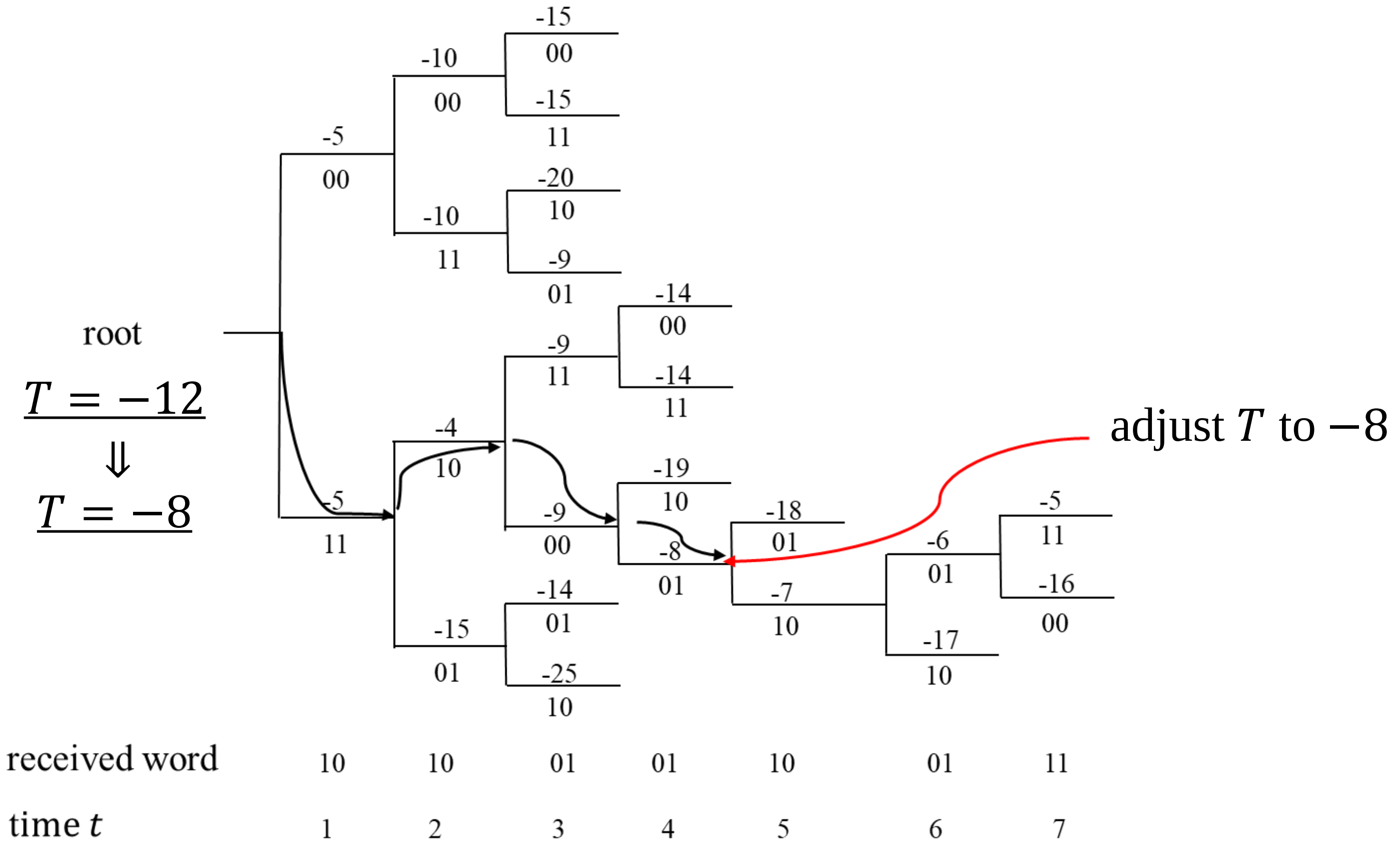
§ 5.4* Sequential Decoding

Exploration of the remaining paths over the tree.



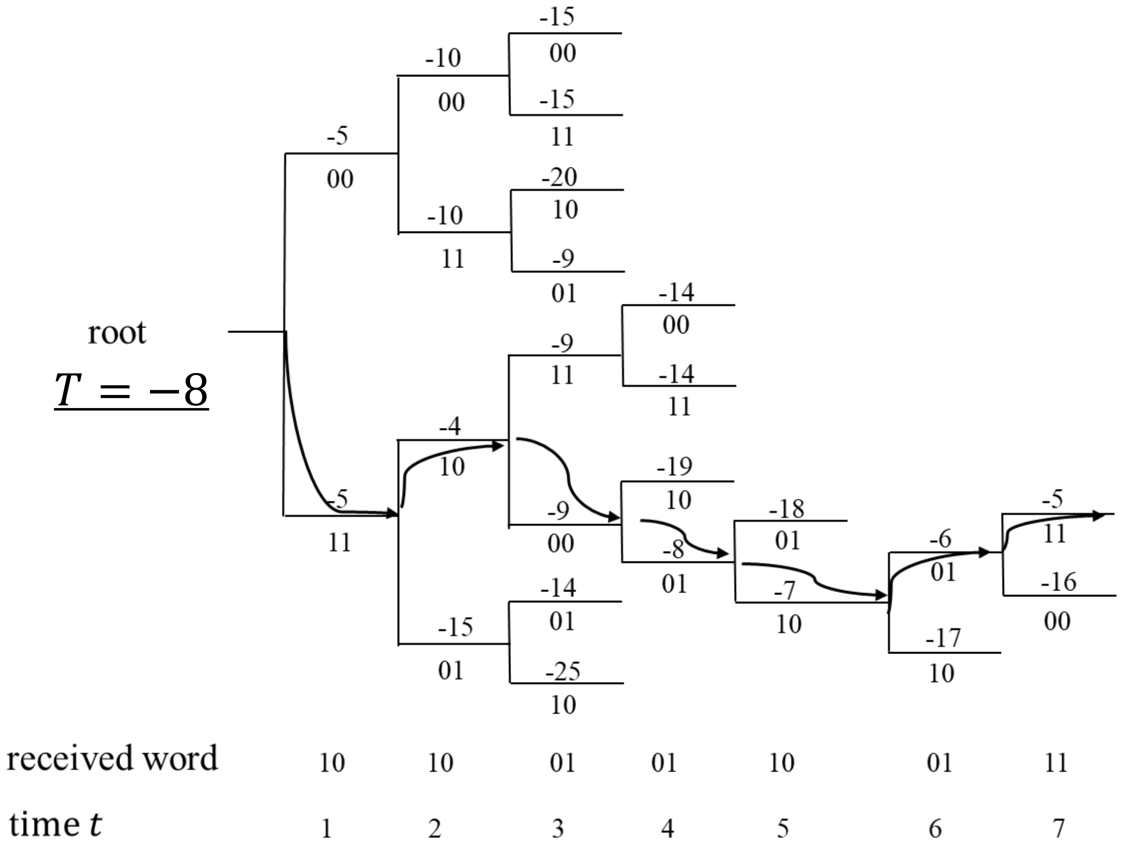
§ 5.4* Sequential Decoding

If reach a node for the first time and its path metric
$\Lambda = -8 \geq T + \Delta = -8$, let $T + \Delta \to T$, s.t. $\Lambda - \Delta < T \leq \Lambda$.



§ 5.4* Sequential Decoding

When decoder reach the leaf node, the decoding is completed and the path $[\hat{m}_1\,\hat{m}_2\,\hat{m}_3\,\hat{m}_4\,\hat{m}_5\,\hat{m}_6\,\hat{m}_7] = [1\ 0\ 1\ 1\ 1\ 0\ 0]$ is output.



§ 5.4* Sequential Decoding

– Parameter selecting:
  - With a small Δ, the search is precise, but it often falls back to the root node to adjust the threshold T.
  - With a large Δ, the searching speed may be fast. But it is easy to explore the wrong path and stuck, causing an increase in complexity.
  - Choose Δ empirically, such as 2 or 4.

– Fano path metric Λ is used for comparing paths of different lengths. The path with the maximum Fano path metric is considered the most reliable one.

– If the path is correct, Λ would increase slightly. If current path deviates from the correct path, Λ would continue to decline when going forward.

– Fano decoder will linger between several paths with similar path metrics, which is the main cause of its latency and complexity. In communication systems, the uncertainty of the duration from the beginning to the end of decoding is often undesirable.